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WAVELET TRANSFORM AND ORTHOGONAL DECOMPOSITION OF L^2 SPACE ON THE CARTAN DOMAIN $BDI(q=2)$

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ABSTRACT. Let $G = \left(\mathbb{R}_+^* \times SO_0(1,n)\right) \ltimes \mathbb{R}^{n+1}$ be the Weyl-Poincaré group and KAN be the Iwasawa decomposition of $SO_0(1,n)$ with $K = SO(n)$. Then the "affine Weyl-Poincaré group" $G_a = \left(\mathbb{R}_+^* \times AN\right) \ltimes \mathbb{R}^{n+1}$ can be realized as the complex tube domain $\Pi = \mathbb{R}^{n+1} + iC$ or the classical Cartan domain $BDI(q=2)$. The square-integrable representations of G and G_a give the admissible wavelets and wavelet transforms. An orthogonal basis $\{\psi_k\}$ of the set of admissible wavelets associated to G_a is constructed, and it gives an orthogonal decomposition of L^2 space on Π (or the Cartan domain $BDI(q=2)$) with every component A_k being the range of wavelet transforms of functions in H^2 with ψ_k.

1. Introduction

The wavelet transform is associated to the square-integrable representation of a locally compact group. Let G be such a group with left Haar measure dx and $x \to U(x)(x \in G)$ be an irreducible unitary representation of G in a Hilbert space $\mathcal{H}$. A vector $\psi \in \mathcal{H}$ is said to be admissible if it satisfies the following "admissibility condition":

$$0 < c_\psi := \int_G |(\psi, U(x)\psi)|^2 dx/(\psi,\psi) < \infty, \tag{1.1}$$

where $(\cdot,\cdot)$ is the inner product of $\mathcal{H}$. We denote the set of all such vectors by AW. If $AW \neq \varnothing$, then the representation U is called square-integrable. For $\psi \in AW$, $f \to (f, U(x)\psi)$ is called the "continuous wavelet transform" of $f \in \mathcal{H}$ (cf. [4], [5]), and

$$f = \frac{1}{c_\psi}\int_G W_\psi f(x) U(x)\psi dx. \tag{1.2}$$

(1.2) is usually called the reconstructing formula, and it is one of the main motivations for the study of the wavelet transform. In this paper we will consider the wavelet transform associated to the Weyl-Poincaré group and its quotient group, and then give an orthogonal decomposition of L^2 space on the Cartan domain $BDI(q=2)$.

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Let $C := \{(x_0, x_1, \cdots, x_n) : x_0^2 - x_1^2 - \cdots - x_n^2 > 0, x_0 > 0\}$ be the forward light cone (the Lorentz cone) and $\Pi := \mathbb{R}^{n+1} + iC$. The complex tube domain Π is an (unbounded) realization of the Cartan domain $BDI(q=2)$ $(SO_0(n,2)/SO(n) \times SO(2))$ (see [13], [15]). Let $G := \big(\mathbb{R}_+^* \times SO_0(1,n)\big) \ltimes \mathbb{R}^{n+1}$ be the Weyl-Poincar*é* group; it is the automorphism group of Π, and G modulo a maximal compact subgroup $SO(n)$ is isomorphic to Π. Motivated by the work on quantization on the Cartan domain $BDI(q=2)$ in [13], [14] and on wavelet transform in [1], we considered in [7] the wavelet transform associated to G and its homogeneous space Π.

Denote

$$H^2 := \{f(y) : \operatorname{supp} \widehat{f} \subset \overline{C}, f \in L^2(\mathbb{R}^{n+1}, dx)\}, \tag{1.3}$$

where $\overline{C}$ is the closure of C. In [7], from the square-integrable representation of the Weyl-Poincar*é* group G on H^2, we got the corresponding admissibility condition:

$$0 < \int_C |\widehat{\psi}(x)|^2 \frac{dx}{r(x)^{\frac{n+1}{2}}} < \infty.$$

For such ψ, (1.2) holds, i.e. for any $f \in H^2$, it is reconstructed from functions $W_\psi f(g)$ on G. In fact it doesn't need so many variables for its reconstruction, which is very important in applications. On the other hand, from the viewpoint of the decomposition of L^2 on $\Pi = \mathbb{R}^{n+1} + iC$ via the wavelet transform, we need a wavelet transform W_ψ such that $W_\psi f(z)$ are functions on Π. Therefore we considered there the wavelet transform associated to the homogeneous space Π (such a wavelet transform was just the one considered in [1]). In order that for any $f \in H^2$ it can be reconstructed from $W_\psi f(z)$, the admissibility condition in this case is that

$$\int_C |\widehat{\psi}(\Lambda_y x)|^2 dy/r(y)^{\frac{n+1}{2}} = \int_C |\widehat{\psi}(y)|^2 dy/r(y)^{\frac{n+1}{2}} \quad \text{for all } x \in C, \tag{1.4}$$

and is finite, where Λ_y is an element in $\mathbb{R}_+^* \times SO_0(1,n)$ such that $\Lambda_y \omega = y$.

The condition in (1.4) is troublesome if we wish to give an orthogonal basis for the set of admissible wavelets which can give an orthogonal decomposition of L^2 on Π (or on the Cartan domain $BDI(q=2)$). In this paper we will consider again the wavelet transform associated to $G/SO(n)$ such that the condition (1.4) can be removed.

In §2 we introduce the wavelet transforms associated to G and to $G/SO(n)$ and a kind of generalized wavelet transform. The main part of this paper is §3 and §4. In §3, we give a correspondence from Π to the quotient ("affine ") group $G_a := \big(\mathbb{R}_+^* \times AN\big) \ltimes \mathbb{R}^{n+1}$ of G from the Iwasawa decomposition of $SO_0(1,n) = KAN$; then from the square-integrable representation of G_a on H^2 we define the associated wavelet transform and get that the admissibility condition is

$$0 < \int_C |\widehat{\psi}(y_0, y_1, y_*)|^2 \frac{dy_0 dy_1 dy_*}{(y_0+y_1)^{n-1} r(y)} < \infty.$$

In §4, by Laguerre polynomials, Jacobi polynomials and spherical harmonics we give an orthogonal basis for the set of the admissible wavelets which turns out to give an orthogonal decomposition of L^2 on Π.

2. Square-integrable group representation

Let ${}^tM$ denote the transpose of a matrix M, and display a vector $x \in \mathbb{R}^{n+1}$ formally as a column vector in the form $x = {}^t(x_0, x^*) = {}^t(x_0, x_1, \cdots, x_n)$. Let I_n

denote the $n \times n$ identity matrix and let

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix} \in GL(\mathbb{R}^{n+1}).$$

For any $x, y \in \mathbb{R}^{n+1}$, let $x \cdot y$ or $[x, y]$ be the bilinear form

$$x \cdot y := [x, y] := {}^t x J y = x_0 y_0 - x_1 y_1 - \cdots - x_n y_n,$$

and $r(x) := x \cdot x$; also let $xy := (x, y) := \sum_{i=0}^{n} x_i y_i$ be the usual Euclidean inner product of $\mathbb{R}^{n+1}$.

The Lorentz group $SO(1, n)$ is the subgroup of $\Lambda \in GL(\mathbb{R}^{n+1})$ satisfying $\det\Lambda = 1$ and ${}^t\Lambda J \Lambda = J$ (equivalently $[\Lambda x, \Lambda y] = [x, y]$ for any $x, y \in \mathbb{R}^{n+1}$). Let $SO_0(1, n)$ denote the connected component of the identity of $SO(1, n)$, and let $G_0 := \mathbb{R}_+^* \times SO_0(1, n)$. Then the Weyl-Poincaré group G is the semi-product $G_0 \ltimes \mathbb{R}^{n+1}$. Write $g \in G$ as $g = (a, \Lambda, b)$ with $a > 0, \Lambda \in SO_0(1, n), b \in \mathbb{R}^{n+1}$ then the group law of G is given by

$$(a, \Lambda, b)(a', \Lambda', b') = (aa', \Lambda\Lambda', a\Lambda b' + b)$$

for any $(a, \Lambda, b), (a', \Lambda', b') \in G$, and one can see the left invariant measure $d\mu(g)$ of G is $a^{-(n+2)} da d\Lambda db$; here $d\Lambda$ is the invariant measure of $SO_0(1, n)$. As in §1 let C denote the forward light cone (the Lorentz cone) and $\Pi = \mathbb{R}^{n+1} + iC$ the complex tube domain over C. The actions of G_0 on C and G on Π are given respectively by

$$(a, \Lambda)y = a\Lambda y, \quad (g, b)(x + iy) = g(x + iy) + b.$$

Then G_0 acts transitively on C, and measures $r(y)^{-\frac{n+1}{2}} dy$ on C and $r(y)^{-(n+1)} dx dy$ on Π are invariant under the actions of G_0 and G respectively; here dy, dx denote the Lebesgue measure on $\mathbb{R}^{n+1}$.

Still let H^2 denote the space defined by (1.3) and let $(\cdot, \cdot)$ or $(\cdot, \cdot)_{H^2}$ denote its inner product. Let U_g be the unitary representation of G on H^2 given by

$$U_g f(x) := a^{-\frac{n+1}{2}} f(\Lambda^{-1} \frac{x - b}{a}). \tag{2.1}$$

Taking the Fourier transform with respect to x in both sides of (2.1), we get

$$(U_g f)^\wedge(\xi) = a^{\frac{n+1}{2}} \widehat{\psi}(a^t\Lambda\xi) e^{-i\xi b},$$

and therefore we know U_g is irreducible on H^2. By the Plancherel formula,

$$\begin{aligned}
&\int_G |(\psi, U_g\psi)|^2 d\mu(g) = \int_G |a^{\frac{n+1}{2}} \int \widehat{\psi}(\xi)\overline{\widehat{\psi}(a^t\Lambda\xi)} e^{i\xi b} d\xi|^2 \frac{da d\Lambda db}{a^{n+2}} \\
&= \int_{\mathbb{R}_+^* \times SO_0(1,n) \times \mathbb{R}^{n+1}} |\widehat{\psi}(\xi)\widehat{\psi}(a^t\Lambda\xi)|^2 \frac{da d\Lambda d\xi}{a} \\
&= \int_C \int_{\mathbb{R}_+^* \times SO_0(1,n)} |\widehat{\psi}(a^t\Lambda\xi)|^2 \frac{da d\Lambda}{a} |\widehat{\psi}(\xi)|^2 d\xi.
\end{aligned}$$

In the above equations, a constant $1/(2\pi)^{n+1}$ is dropped. In the following, for simplicity, we also drop this constant in some equations when we use the Plancherel formula.

For $\xi \in C$, there exists $\Lambda_\xi \in SO_0(1, n)$ such that $\Lambda_\xi \omega = \xi / r(\xi)^{\frac{1}{2}}$ (see [16, p.505]); here

$$\omega := (1, 0, \cdots, 0).$$

By the left invariant property of $d\Lambda$ and the fact that ${}^t\Lambda_\xi = J\Lambda_\xi^{-1}J \in SO_0(1,n)$, we have

$$\int_{\mathbb{R}_+^* \times SO_0(1,n)} |\widehat{\psi}(a^t\Lambda\xi)|^2 \frac{dad\Lambda}{a} = \int_0^\infty \int_{SO_0(1,n)} |\widehat{\psi}(a^t\Lambda\omega)|^2 \frac{d\Lambda da}{a}.$$

From the "Euler angles" expression of $\Lambda \in SO_0(1,n)$ as given in [16, p.508] and the "spherical coordinates" expression of $x \in C$ as given in [16, p.504], we can get that for any $\xi \in C$

$$\int_{\mathbb{R}_+^* \times SO_0(1,n)} |\widehat{\psi}(a^t\Lambda\xi)|^2 \frac{d\Lambda da}{a} = \int_0^\infty \int_{SO_0(1,n)} |\widehat{\psi}(aJ\Lambda^{-1}J\omega)|^2 \frac{dad\Lambda}{a}$$
$$= \int_C |\widehat{\psi}(x)|^2 \frac{dx}{r(x)^{\frac{n+1}{2}}}.$$

Therefore the admissibility condition is

$$0 < c_\psi = \int_C |\widehat{\psi}(x)|^2 \frac{dx}{r(x)^{\frac{n+1}{2}}} < \infty. \tag{2.2}$$

Let AW denote the set of all admissible wavelets, i.e.

$$AW := \{\psi : \psi \in H^2,\ \psi \text{ satisfies (2.2)}\}.$$

For $\alpha > -1$, let ψ be a function in H^2 defined by

$$\widehat{\psi}(\xi) = \begin{cases} r(\xi)^\alpha e^{-\xi_0}, & \text{for } \xi \in C, \\ 0, & \text{elsewhere;} \end{cases} \tag{2.3}$$

then $\psi \in AW$ by a direct calculation. Thus the unitary irreducible representation given by (2.1) is square-integrable. For $\psi \in AW$, the map $f \to W_\psi f(g) := (f, U_g\psi)$ is the wavelet transform, and it is an isometry (up to a constant) from H^2 into $L^2(G, d\mu(g))$. One can get

Theorem 2.1. *Let $\psi, \phi \in AW$. Then for any $f, h \in H^2$,*

$$\langle W_\psi f, W_\phi h\rangle = (K^{-\frac{1}{2}}\phi, K^{-\frac{1}{2}}\psi)(f, h),$$

where $\langle , \rangle$ is the inner product of $L^2(G, d\mu(g)$ and K is the positive operator given by

$$(K\psi)^\wedge(\xi) = r(\xi)^{\frac{n+1}{2}}\widehat{\psi}(\xi).$$

Theorem 2.2. *Let $\psi \in AW$. Then for any $f \in H^2$,*

$$f(x) = \frac{1}{c_\psi}\int_G W_\psi f(g) U_g\psi(x) d\mu(g).$$

For $\psi \in AW$, it is a (generalized) state in H^2 which can be written as $\psi\rangle$. Then $\{U_g\psi\}_{g\in G} = \{U_g|\psi\rangle\}_{g\in G}$ is a coherent state system [9]. Denote

$$A_\psi := \{\langle f|U_g|\psi\rangle : f \in H^2\} = \{W_\psi f(g) : f \in H^2\},$$

i.e. A_ψ is the matrix coefficient space of the representation U_g or the range of the wavelet transform W_ψ of H^2 with ψ. Then A_ψ is a Hilbert space with reproducing kernel.

Theorem 2.3. *Let $K_\psi(g,g')$ denote the reproducing kernel of A_ψ. Then*

$$K_\psi(g,g') = \frac{1}{c_\psi}(U_{g'}\psi, U_g\psi).$$

Let $SO(n)$ denote the group of rotations of $\mathbb{R}^n$ and in this paper also let it denote the subgroup of $SO_0(1,n)$ fixing the point ω, i.e. every element u of this subgroup is given by

$$u = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{u} \end{pmatrix}, \quad \text{with } \tilde{u} \in SO(n).$$

We even let $SO(n)$ denote the subsgroup $(\{1\} \times SO(n)) \ltimes \{0\}$ of G. Then $SO(n)$ is a maximal subgroup of G and the quotient space $G/SO(n)$ is isomorphic to the homogeneous space Π. From Theorem 2.2, we know for any $f \in H^2$ it is reconstructed from $W_\psi f(g)$. The wavelet transform $W_\psi f(g)$ is a function of $n+2+\frac{n(n+1)}{2}$ variables. In fact f doesn't need so many variables for its reconstruction. We consider the wavelet transform associated to $G/SO(n)$ or to Π by a correspondence from Π to $G/SO(n)$.

Recall that the action of $g = (a,\Lambda,b) \in G$ on Π is given by

$$(a,\Lambda,b)z = a\Lambda z + b = a\Lambda(x+iy) + b.$$

Denote $z_0 := 0 + i\omega \in \Pi$. For $z = x+iy \in \Pi$, there exist a family of $g \in G$ such that $gz_0 = z$. In the following we choose $g = (r(y)^{\frac{1}{2}}, \Lambda_{y'}, x)$, where $\Lambda_{y'} = r(y)^{-\frac{1}{2}}\Lambda_y$ and Λ_y is given in [14]:

$$\Lambda_y := \begin{pmatrix} y_0 & {}^t y^* \\ y^* & r(y)^{\frac{1}{2}} I + (y_0 + r(y)^{\frac{1}{2}})^{-1} y^* \, {}^t y^* \end{pmatrix}, \qquad \text{with } y = (y_0, y^*). \tag{2.4}$$

Then ${}^t\Lambda_y = \Lambda_y, \Lambda_y\omega = y$ and $\Lambda_{y'} \in SO_0(1,n), \Lambda_y \in G_0$. For each $z \in \Pi$, from (2.1), we define an operator U_z on H^2 by

$$U_z\psi(t) := r(y)^{-\frac{n+1}{4}}\psi\left(r(y)^{-\frac{1}{2}}\Lambda_{y'}^{-1}(t-x)\right)$$

or by

$$(U_z\psi)^\wedge(\xi) := r(y)^{\frac{n+1}{4}}\widehat{\psi}\left(r(y)^{\frac{1}{2}}\, {}^t\Lambda_{y'}\xi\right)e^{-ix\xi} = r(y)^{\frac{n+1}{4}}\widehat{\psi}(\Lambda_y\xi)e^{-ix\xi}, \tag{2.5}$$

since ${}^t\Lambda_y = \Lambda_y$ from (2.4). Then we define the associated wavelet transform for $f \in H^2$ by

$$W_\psi f(z) := (f, U_z\psi) \tag{2.6}$$

or by

$$(W_\psi f)^\wedge(\xi, y) := r(y)^{\frac{n+1}{4}}\overline{\widehat{\psi}(\Lambda_y\xi)}\widehat{f}(\xi), \tag{2.7}$$

where $(W_\psi f)^\wedge(\xi,y)$ denotes the Fourier transform of $W_\psi f(z)$ with respect to the variables x.

Any $f \in H^2$ also can be reconstructed from the wavelet transform $W_\psi f(z)$ for some ψ, i.e. the following formula holds:

$$f(\tau) = c\int_\Pi W_\psi f(z) U_z\psi(\tau)d\mu(z), \tag{2.8}$$

where $d\mu(z) = dxdy/r(y)^{n+1}$ is the invariant measure on Π under G. Let us give the corresponding admissibility condition for ψ. By the Plancherel formula

$$\begin{aligned}\int_\Pi |W_\psi\psi(z)|^2 d\mu(z) &= \int_\Pi |\widehat{\psi}(\xi)\hat{\psi}(\Lambda_y\xi)|^2 d\xi \frac{dy}{r(y)^{\frac{n+1}{2}}}\\ &= \int_C |\widehat{\psi}(\xi)|^2 \int_C |\hat{\psi}(\Lambda_y\xi)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}} d\xi.\end{aligned}$$

Thus in order that (2.8) holds, ψ must be such that $\int_C |\hat{\psi}(\Lambda_y\xi)|^2 dy/r(y)^{\frac{n+1}{2}}$ is independent of all $\xi \in C$; especially, if $\xi = \omega$, then it is $\int_C |\hat{\psi}(y)|^2 dy/r(y)^{\frac{n+1}{2}}$. Thus the admissibility condition in this case is that ψ satisfies (2.2) and

$$\int_C |\widehat{\psi}(\Lambda_y\xi)|^2 dy/r(y)^{\frac{n+1}{2}} = \int_C |\widehat{\psi}(y)|^2 dy/r(y)^{\frac{n+1}{2}} \quad \text{for all } \xi \in C. \tag{2.9}$$

If $n > 1$, (2.2) does not imply (2.9), i.e. there are many functions $\psi \in H^2$ satisfying (2.2) but not (2.9). If ψ is invariant under the action of $SO(n)$ or "radial", i.e. $\psi(x_0, \rho x^*) = \psi(x_0, x^*)$ for all $\rho \in SO(n)$, then it satisfies (2.9). In fact, in this case, $\widehat{\psi}(\xi)$ is also "radial" and can be written as

$$\widehat{\psi}(\xi) = \phi(\xi_0, |\xi^*|^2) = \phi\left((\xi,\omega), (\xi,\omega)^2 - r(\xi)\right),$$

where $(\cdot,\cdot)$ is the usual Euclidean inner product of $\mathbb{R}^{n+1}$ as mentioned above. For $\xi, y \in C$, let $\Lambda_\xi, \Lambda_y \in G_0 = \mathbb{R}_+^* \times SO_0(1,n)$ be defined by (2.4). Then

$$(\Lambda_y\xi, \omega) = (\xi, \Lambda_y\omega) = (\Lambda_\xi\omega, y) = (\omega, \Lambda_\xi y),$$

since ${}^t\Lambda_y = \Lambda_y, {}^t\Lambda_\xi = \Lambda_\xi$, and we have

$$\begin{aligned}\int_C |\widehat{\psi}(\Lambda_y\xi)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}} &= \int_C |\phi\left((\Lambda_y\xi,\omega), (\Lambda_y\xi,\omega)^2 - r(\Lambda_y\xi)\right)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}}\\ &= \int_C |\phi\left((\omega,\Lambda_\xi y), (\omega,\Lambda_\xi y)^2 - r(y)r(\xi)\right)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}}\\ &= \int_C |\phi\left((\omega,y), (\omega,y)^2 - r(y)\right)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}}\\ &= \int_C |\phi(y_0, y_0^2 - r(y))|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}} = \int_C |\widehat{\psi}(y)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}}.\end{aligned}$$

The third equality is from the invariant property of $dy/r(y)^{\frac{n+1}{2}}$ under the action of G_0 and the fact that $r(y)r(\xi) = r(\Lambda_\xi y)$. Thus when ψ is "radial", the admissibility condition for (2.8) is (2.2).

For ψ satisfying (2.2) and (2.9), define the wavelet transform for $f \in H^2$ via (2.6); then for such a wavelet transform we can establish theorems similar to Theorems 2.1, 2.2, 2.3, but we won't bother to list them here.

Condition (2.9) is troublesome if we want to give an orthogonal basis for the set of admissible wavelets. We now introduce, as in [8], a kind of generalized wavelet transform associated to Π. In this case the admissibility condition is (2.2) and (2.9) will be removed.

For each $z \in \Pi$, let us introduce an operator $\widetilde{U}_z$ on H^2 similar to U_z defined above. For $z \in \Pi$, define $\widetilde{U}_z$ on H^2 by (compare with the definition of U_z given in (2.5))

$$(\widetilde{U}_z\psi)^\wedge(\xi) := r(y)^{\frac{n+1}{4}}\widehat{\psi}(\Lambda_\xi y)e^{-ix\xi}, \tag{2.10}$$

where $\psi \in H^2$ and Λ_ξ is given by (2.4) corresponding to $\xi \in C$. Then from (2.10) we define a kind of generalized wavelet transform $\widetilde{W}_\psi$ of functions f in H^2 with some ψ by

$$\widetilde{W}_\psi f(z) := (f, \widetilde{U}_z\psi),$$

or by (compare with (2.7))

$$(\widetilde{W}_\psi f)^\wedge(\xi, y) := r(y)^{\frac{n+1}{4}}\overline{\widehat{\psi}(\Lambda_\xi y)}\widehat{f}(\xi).$$

Let us give the condition on ψ so that for any $f \in H^2$ it can be reconstructed from $\widetilde{W}_\psi f(z)$. Of course, $\widetilde{W}_\psi$ must satisfy $\int_\Pi |\widetilde{W}_\psi f|^2 d\mu(z) = C\|f\|^2$, where $d\mu(z)$ is the invariant measure $dxdy/r(y)^{n+1}$. We have

$$\begin{aligned}\int_\Pi |\widetilde{W}_\psi f|^2 d\mu(z) &= \int_\Pi |\widehat{\psi}(\Lambda_\xi y)\widehat{f}(\xi)|^2 d\xi \frac{dy}{r(y)^{\frac{n+1}{2}}} \\ &= \int_C \int_C |\widehat{\psi}(\Lambda_\xi y)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}} |\widehat{f}(\xi)|^2 d\xi.\end{aligned}$$

Then by the invariant property of $dy/r(y)^{\frac{n+1}{2}}$ under actions of G_0, we have

$$\int_C |\widehat{\psi}(\Lambda_\xi y)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}} = \int_C |\widehat{\psi}(r(\xi)^{\frac{1}{2}}\Lambda_{\xi'} y)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}} = \int_C |\widehat{\psi}(y)|^2 \frac{dy}{r(y)^{\frac{n+1}{2}}}.$$

Thus in this case, the admissibility condition is $\int_C |\hat{\psi}(y)|^2 dy/r(y)^{\frac{n+1}{2}} < \infty$. Let AW denote the set of all the admissible wavelets, i.e. AW consists of the ψ satisfying (2.2). Then for $\psi \in AW$, we can prove that any $f \in H^2$ can also be reconstructed from $\widetilde{W}_\psi f(z)$.

Though it is easy to check the admissibility condition for this kind of generalized wavelet transform and to give an orthogonal basis for AW via Laguerre polynomials, Jacobi polynomials and spherical harmonics which will give an orthogonal decomposition of L^2 on Π, still such a wavelet transform does not associate to the square-integrable group representation and looks artificial. In next section, we will introduce another kind of wavelet transform from the square-integrable representation of a quotient group of the Weyl-Poincaré group.

3. Square-integrable representation modulo a subgroup

In this section we will introduce a quotient group of the Weyl-Poincaré group $G = (\mathbb{R}_+^* \times SO_0(1,n)) \ltimes \mathbb{R}^{n+1}$ and a kind of wavelet transform from the square-integrable representation of this group. Let $K = SO(n)$ be the rotation group of $\mathbb{R}^n$; as in §2 it also denotes a subgroup of $SO_0(1,n)$ and even a subgroup of G. Then K is a maximal compact subgroup of G. The quotient group G/K can realized as the complex tube domain Π via the Iwasawa decomposition of G (in fact the decomposition of $SO_0(1,n)$).

Let g be the Lie algebra of $SO_0(1,n)$; then (see [12, p.222])

$$g = l \oplus a \oplus n,$$

where l is the Lie algebra of $K = SO(n)$, a is an one-dimensional algebra with generator

$$a_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(n+1)\times(n+1)},$$

and

$$n = \{x = \begin{pmatrix} 0 & 0 & {}^ts \\ 0 & 0 & {}^ts \\ s & -s & O_{n-1} \end{pmatrix} : \quad s = {}^t(s_2, \cdots, s_n) \in \mathbb{R}^{n-1}\}.$$

Then $SO_0(1, n)$ has the Iwasawa decomposition

$$SO_0(1, n) = KAN$$

with

$$A = \{a_r := \begin{pmatrix} chr & shr & 0 \\ shr & chr & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} : \quad r \in \mathbb{R}\},$$

$$N = \{n_s := \begin{pmatrix} \rho + 1 & -\rho & {}^ts \\ \rho & 1 - \rho & {}^ts \\ s & -s & I_{n-1} \end{pmatrix} : \quad s = {}^t(s_2, \cdots, s_n) \in \mathbb{R}^{n-1}, \rho = \frac{1}{2}|s|^2\}.$$

The action of $a_r \in A$ on N is given

$$a_r : \quad n_s \to a_r n_s a_r^{-1} = n_{e^r s}.$$

By a direct calculation, we have $a_r a_{r'} = a_{r+r'}, n_s n_{s'} = n_{s+s'}$ and

$$(3.1) \qquad (a_r, n_s)(a_{r'}, n_{s'}) = a_{r+r'} n_{s'+se^{-r'}}$$

for any $a_r, a_{r'} \in A, n_s, n_{s'} \in N$.

Let $H_1 := \{x \in C : r(x) = 1\}$, the forward mass hyperboloid. We know $SO_0(1, n)/K = AN$ is isomorphic to H_1, and for $y' \in H_1$ there exists a unique $a_r n_s \in AN$ such that

$$a_r n_s \omega = y';$$

also, we have

$$(3.2) \qquad chr + e^r \rho = y'_0, \quad shr + e^r \rho = y'_1, \quad s = {}^t(y'_2, \cdots, y'_n).$$

In this way we give a correspondence between AN and H_1 by

$$(3.3) \qquad y' = {}^t(y'_0, y'_1, y'_*) \in H_1 \leftrightarrow a_r n_s \in AN \quad \text{with } e^{-r} = y'_0 - y'_1, \quad s = y'_*.$$

Denote

$$(3.4) \qquad \Lambda_{r,s} := a_r n_s = \begin{pmatrix} chr + e^r \rho & shr - e^r \rho & {}^tse^r \\ shr + e^r \rho & chr - e^r \rho & {}^tse^r \\ s & -s & I_{n-1} \end{pmatrix};$$

then $\Lambda_{r,s} \in SO_0(1, n), \Lambda_{r,s}\Lambda_{r',s'} = \Lambda_{r+r',s'+se^{-r'}}, \Lambda_{r,s}^{-1} = \Lambda_{-r,-se^r}$ and $\Lambda_{r,s}\omega = y'$. From (3.1), let us introduce a group, the "affine Weyl-Poincaré group",

$$G_a := \{(a, r, s, x) | a > 0, r \in \mathbb{R}, s \in \mathbb{R}^{n-1}, x \in \mathbb{R}^{n+1}\}$$

with group law

$$(a,r,s,x)(a',r',s',x') = (aa', r+r', s'+se^{-r'}, x+a\Lambda_{r,s}x').$$

We have $g^{-1} = (a,r,s,x)^{-1} = (\frac{1}{a}, -r, -se^r, -\frac{1}{a}\Lambda_{r,s}^{-1}x)$, and G_a has the left invariant measure

$$d\mu(g) = a^{-(n+2)} dadrdsdx. \tag{3.5}$$

The group G_a defined above is a quotient group $\left(\mathbb{R}_+^* \times SO_0(1,n)\right) \ltimes \mathbb{R}^{n+1}/K$ of G, and it is isomorphic to Π by the following correspondence:

$$z = x+iy \in \Pi \leftrightarrow (a,r,s,x) \in G_a, \tag{3.6}$$

with

$$a = r(y)^{\frac{1}{2}}, \quad e^{-r} = r(y)^{-\frac{1}{2}}(y_0 - y_1), \quad s = r(y)^{-\frac{1}{2}} y_* \tag{3.7}$$

and $\Lambda_{r,s}\omega = y/r(y)^{\frac{1}{2}}$ (here $\Lambda_{r,s}$ is given by (3.4)).

As usual, for $g \in G_a$, with $g^{-1} = \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}$, $B \in \mathbb{R}^{n+1}$, $A \in \mathbb{R}_+^* \times SO_0(1,n)$, g acts on $\mathbb{R}^{n+1}$ by $g(\tau) = A\tau + B$. For $g = (a,r,s,x)$, g^{-1} can be written as

$$\begin{pmatrix} \frac{1}{a}\Lambda_{r,s}^{-1} & -\frac{1}{a}\Lambda_{r,s}^{-1}x \\ 0 & 1 \end{pmatrix},$$

and thus we have

$$g(\tau) := \frac{1}{a}\Lambda_{r,s}^{-1}\tau - \frac{1}{a}\Lambda_{r,s}^{-1}x = \frac{1}{a}\Lambda_{r,s}^{-1}(\tau - x).$$

The above action of $g \in G_a$ on $\mathbb{R}^{n+1}$ induces a unitary representation of G_a on H^2, still denoted by U_g:

$$U_g f(\tau) := \{g'(\tau)\}^{\frac{1}{2}} f\left(g(\tau)\right) = a^{-\frac{n+1}{2}} f\left(a^{-1}\Lambda_{r,s}^{-1}(\tau - x)\right). \tag{3.8}$$

Taking the Fourier transform with respect to τ in both sides of (3.8), we get

$$\begin{aligned}
(U_g f)^\wedge(\xi) &= a^{\frac{n+1}{2}} \int f(\Lambda_{r,s}^{-1}\tau) e^{-ia\xi\tau} d\tau e^{-i\xi x} \\
&= a^{\frac{n+1}{2}} \int f(\Lambda_{r,s}^{-1}\tau) e^{-ia({}^t\Lambda_{r,s}\xi)\Lambda_{r,s}^{-1}\tau} d\tau e^{-i\xi x} = a^{\frac{n+1}{2}} \widehat{f}(a\,{}^t\Lambda_{r,s}\xi) e^{-i\xi x};
\end{aligned}$$

thus we know U_g is irreducible on H^2. We can get the admissibility condition (see the Appendix):

$$0 < C_\psi := \int_0^\infty \int_R \int_{R^n} |\widehat{\psi}(y_0, y_1, y_*)|^2 \frac{dy_0 dy_1 dy_*}{(y_0 + y_1)^{n-1} r(y)} < +\infty. \tag{3.9}$$

When $n = 1$, (3.9) is just (2.2). Let

$$AAW := \{\psi : \psi \in H^2,\ \psi \text{ satisfies } (3.9)\}.$$

Then by a direct calculation, we know the function ψ defined by (2.3) is in AAW; thus the representation of G_a given by (3.8) is square-integrable on H^2. Let $\psi \in AAW$, and define the wavelet transform for $f \in H^2$ from the square-integrable representation of G_a by

$$f \to (f, U_g\psi). \tag{3.10}$$

For such a wavelet transform, we can establish theorems similar to 2.1, 2.2 and 2.3; here the invariant measure of G_a is $d\mu(g)$ given in (3.5). We wouldn't list them here.

By the correspondence given by (3.6) and (3.7), for each $z \in \Pi$, we define U_z by (3.8), i.e.

$$U_z f(\tau) := r(y)^{-\frac{n+1}{4}} f\left(r(y)^{-\frac{1}{2}}\Lambda_{r,s}^{-1}(\tau - x)\right),$$

where r, s are given by (3.7) and $\Lambda_{r,s} = a_r n_s$ by (3.4). Then for any fixed $\psi \in AAW$, or state $\psi\rangle$, applying U_z to it, we get a coherent system $\{U_z|\psi\rangle\}_{z\in\Pi}$ [9]. Wavelet transforms of functions $f \in H^2$ defined by (3.10) are functions on G_a. We now define the wavelet transform via U_z :

$$W_\psi f(z) := \langle f|U_z|\psi\rangle = (f, U_z\psi), \quad \text{for } f \in H^2, \tag{3.11}$$

then $W_\psi f(z)$ are functions on Π, and $f \in H^2$ can be reconstructed from $W_\psi f(z)$. Let $d\mu(z)$ be the invariant measure on Π:

$$d\mu(z) := dxdy/r(y)^{n+1}, \quad \text{with } z = x + iy;$$

then we have

Theorem 3.1. *Let $\psi, \phi \in AAW$. Then for any $f, h \in H^2$,*

$$\langle W_\psi f, W_\phi h\rangle = (K^{-\frac{1}{2}}\phi, K^{-\frac{1}{2}}\psi)(f, h), \tag{3.12}$$

where $\langle,\rangle$ is the inner product of $L^2(\Pi, d\mu(z))$ and K is the positive operator given by

$$(Kf)^\wedge(\xi) = (\xi_0 + \xi_1)^{n-1} r(\xi)\widehat{f}(\xi).$$

Theorem 3.2. *Let $\psi \in AAW$. Then for any $f \in H^2$,*

$$f(\tau) = \frac{1}{C_\psi}\int_\Pi W_\psi f(z) U_z\psi(\tau) d\mu(z). \tag{3.13}$$

For $\psi \in AAW$, denote

$$A_\psi := \{\langle f|U_z|\psi\rangle : f \in H^2\} = \{W_\psi f(z) : f \in H^2\}. \tag{3.14}$$

Then A_ψ is a subspace of $L^2(\Pi, d\mu(z))$ with a reproducing kernel.

Theorem 3.3. *Let $K_\psi(z, z')$ denote the reproducing kernel of A_ψ. Then*

$$K_\psi(z, z') = \frac{1}{C_\psi}(U_{z'}\psi, U_z\psi).$$

In Theorem 3.1, (3.12) is also true for any $\psi, \phi \in H^2$ satisfying $(K^{-\frac{1}{2}}\psi, K^{-\frac{1}{2}}\phi) < \infty$. Clearly if $\psi, \phi \in AAW$, then $(K^{-\frac{1}{2}}\psi, K^{-\frac{1}{2}}\phi) < \infty$. With the identification (3.6) and (3.7), the tube domain Π inherits the G_a group structure and (3.12) is in fact the Moyal formula; see [2]. For completeness, the proof of Theorem 3.1 will be also given in the Appendix.

In Theorem 3.2, (3.13) is true at least "in the weak sense", i.e., taking the inner product of both sides of (3.13) with any $g \in H^2$ and commuting the inner product with the integral over z in the right hand side leads to a true formula, which is

(3.12) with $\phi = \psi$. The convergence of the integral in (3.13) also holds in the following "strong sense":

(3.15)

$$\lim_{\delta_i \to 0, A, B \to +\infty} \|f(\tau) - C_\psi^{-1} \int_{|x|<A, \delta_1<y_0<B, \delta_2<r(y)/y_0^2} W_\psi f(z) U_z \psi(\tau) d\mu(z)\|_2 = 0.$$

Here the integral stands for the unique element in H^2 that has inner product with $g \in H^2$ given by

$$\int_{|x|<A, \delta_1<y_0<B, \delta_2<r(y)/y_0^2} W_\psi f(z)(U_z\psi, g) d\mu(z);$$

since the absolute value of this is bounded by

$$\begin{aligned} &\int_{|x|<A, \delta_1<y_0<B, \delta_2<r(y)/y_0^2} \|f\|_2 \|U_z\psi\|_2^2 \|g\|_2 \frac{dxdy}{r(y)^{n+1}} \\ &\le c_n A^{n+1} \|f\|_2 \|g\|_2 \|\psi\|_2^2 \int_{\delta_1<y_0<B} \int_{r \le \sqrt{1-\delta_2} y_0} \frac{r^{n-1}}{(y_0^2 - r^2)^{n+1}} dr dy_0 \\ &\le \frac{c_n}{n+1} \frac{A^{n+1}}{\delta_2^{n+1}} \left(\frac{1}{\delta_1^{n+1}} - \frac{1}{B^{n+1}} \right) \|f\|_2 \|g\|_2 \|\psi\|_2^2, \end{aligned}$$

where c_n is a constant, the integral in (3.15) stands for a function in H^2 by the Riesz lemma. The proof of (3.15) will be given in the Appendix. Under some conditions on the decay properties at infinity of $\psi, \widehat{\psi}$, then for bounded $f(x) \in H^2$, (3.13) holds pointwise at every point x where f is continuous. We will not give the details on them here.

The reproducing kernel in Theorem 3.3 can be gotten from Theorem 3.2. In fact, from Theorem 3.2,

$$\begin{aligned} W_\psi f(z) = (f, U_z\psi) &= \frac{1}{C_\psi} \int_\Pi W_\psi f(z')(U_{z'}\psi, U_z\psi) d\mu(z') \\ &= \frac{1}{C_\psi} \int_\Pi (U_{z'}\psi, U_z\psi) W_\psi f(z') d\mu(z'). \end{aligned}$$

Thus $K_\psi(z, z') = \frac{1}{C_\psi}(U_{z'}\psi, U_z\psi)$.

In the next section, we will give an orthogonal decomposition of L^2 on Π by giving an orthogonal basis of AAW.

4. Orthogonal decomposition of L^2 space on $BDI(q = 2)$

In this section we will give an orthogonal decomposition of the space L^2 on Π (or on the Cartan domain $BDI(q = 2)$) in the form $\bigoplus_{\vec{k}} A_{\vec{k}}$, in which the $A_{\vec{k}}$ are the ranges of wavelet transforms defined as in (3.14). First let us show that the Bergman space is such a range with a special choice of admissible wavelet.

For $\alpha \in \mathbb{R}$, let $L^{\alpha 2}(\Pi)$ denote the L^2 space on Π defined by

$$L^{\alpha 2}(\Pi) := \{F(x, y) : \int_\Pi |F(x, y)|^2 r(y)^\alpha dxdy < \infty\}.$$

If $\alpha > -1$, then $L^{\alpha 2}(\Pi)$ contains holomorphic functions. In fact, $\alpha > -1$ is also a necessary condition for the existence of holomorphic functions in $L^{\alpha 2}(\Pi)$; see [3, p.260]. In this paper, we will assume $\alpha > -1$. Let $A^{\alpha 2}$ denote the subspace of $L^{\alpha 2}(\Pi)$ consisting of holomorphic functions; $A^{\alpha 2}$ is called the (weighted) Bergman space. Let $H^{\alpha 2}(\Pi)$ denote the subspace of $L^{\alpha 2}(\Pi)$ of functions $F(x,y)$ such that $\mathrm{supp}\widehat{F}(\xi,y) \subset \overline{C} \times \overline{C}$; here $\widehat{F}(\xi,y)$ denotes the Fourier transform of $F(x,y)$ with respect to the variables x and $\overline{C}$ is the closure of C. Since the cone C is homogeneous and the dual of C is still itself, i.e. C is a regular (symmetric) cone, then for every $F(z) \in A^{\alpha 2}$, there exists an $f \in H^2$ such that (refer to Theorem XIII 1.1 in [3])

$$F(z) = c \int_C e^{i\xi z} \widehat{f}(\xi) r(\xi)^{\frac{2\alpha+n+1}{4}} d\xi, \tag{4.1}$$

where $z = x + iy \in \Pi$. Conversely, for every $f \in H^2, F(z)$, defined by (4.1), belongs to $A^{\alpha 2}$. That is, the map $f \to F$ is a linear isomorphism from H^2 onto $A^{\alpha 2}$. From (4.1), we know if $F(z) \in A^{\alpha 2}$, then $\mathrm{supp}\widehat{F}(\xi,y) \subset \overline{C} \times \overline{C}$, and therefore $A^{\alpha 2} \subset H^{\alpha 2}(\Pi)$.

For $\psi \in AAW$, let $W_\psi f(z)$ be the wavelet transform defined by (3.11); by Theorem 3.1, it is an isometry (up to a constant) from H^2 into $L^2(\Pi, d\mu(z))$. Define

$$W_\psi^\alpha f(z) := r(y)^{-\frac{n+1+\alpha}{2}} W_\psi f(z) = r(y)^{-\frac{n+1+\alpha}{2}} (f, U_z\psi); \tag{4.2}$$

then W_ψ^α is an isometry (up to a constant) from H^2 into $L^{\alpha 2}(\Pi)$, i.e.

$$\int_\Pi |W_\psi^\alpha f(z)|^2 r(y)^\alpha dxdy = C_\psi \|f\|_2^2.$$

About this transform W_ψ^α, we also have theorems similar to Theorem 3.1, 3.2, 3.3. For example, for any $\psi, \phi \in AAW$ and $f, h \in H^2$,

$$\langle W_\psi^\alpha f, W_\psi^\alpha h\rangle_{L^{\alpha 2}(\Pi)} = (f,h) \int_C \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} \frac{d\xi_0 d\xi_1 d\xi_*}{(\xi_0+\xi_1)^{n-1} r(\xi)}, \tag{4.3}$$

which is just (3.12). For $\psi \in AAW$, let A_ψ^α be the space defined by

$$A_\psi^\alpha := \{W_\psi^\alpha f(z) = r(y)^{-\frac{n+1+\alpha}{2}} (f, U_z\psi) : f \in H^2\}. \tag{4.4}$$

Since W_ψ^α is an isometry (up to a constant) from H^2 into $L^{\alpha 2}(\Pi)$, we have $A_\psi^\alpha \subset L^{\alpha 2}(\Pi)$. By (4.2),

$$\left(W_\psi^\alpha f\right)^\wedge (\xi,y) = r(y)^{-\frac{n+1+2\alpha}{4}} \widehat{f}(\xi)\overline{\widehat{\psi}\left(r(y)^{\frac{1}{2}}\, {}^t\Lambda_{r,s}\xi\right)} \quad (\text{with } \Lambda_{r,s}\omega = y/r(y)^{\frac{1}{2}}),$$

and we know that $\mathrm{supp}\left(W_\psi^\alpha f\right)^\wedge (\xi,y) \subset \overline{C} \times \overline{C}$ since $\mathrm{supp}\widehat{f} \subset \overline{C}$. Thus A_ψ^α is a subspace of $H^{\alpha 2}(\Pi)$. We can see that A_ψ^α has a reproducing kernel, given by

$$K_\psi^\alpha(z,z') = C_\psi^{-1}(r(y)r(y'))^{-\frac{n+1+\alpha}{2}} (U_{z'}\psi, U_z\psi), \tag{4.5}$$

with $z = x+iy, z' = x'+iy' \in \Pi$.

Let $\psi_0^\alpha \in AAW$ be defined by

$$\widehat{\psi_0^\alpha}(\xi) = \begin{cases} r(\xi)^{\frac{2\alpha+n+1}{4}} e^{-\xi_0}, & \text{for } \xi \in C, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.6}$$

Then for all $f \in H^2$, the $W_\psi^\alpha f(z)$ are holomorphic on Π. In fact, let $\Lambda_{r,s}$ be the correspondence to $y \in C$ given by $\Lambda_{r,s}\omega = y/r(y)^{\frac{1}{2}}$; then $(\omega, {}^t\Lambda_{r,s}\xi) = (\Lambda_{r,s}\omega, \xi) =$

$y\xi/r(y)^{\frac{1}{2}}$. By the Plancherel formula,

$$W^\alpha_{\psi^\alpha_0} f(z) = r(y)^{-(\alpha+n+1)/2} \int_C r(y)^{\frac{n+1}{4}} \widehat{f}(\xi)\widehat{\psi}^\alpha_0(r(y)^{\frac{1}{2}\,t}\Lambda_{r,s}\xi)e^{ix\xi}d\xi$$

$$= \int_C \widehat{f}(\xi)r(\xi)^{(2\alpha+n+1)/4}e^{ix\xi-y\xi}d\xi = \int_C \widehat{f}(\xi)r(\xi)^{(2\alpha+n+1)/4}e^{i\xi z}d\xi; \tag{4.7}$$

thus $W^\alpha_{\psi^\alpha_0} f(z)$ is holomorphic on Π. From (4.7) and the relation between $A^{\alpha 2}$ and H^2 given by (4.1) (i.e. Theorem XIII 1.1 in [3]), we know $A_{\psi^\alpha_0}$ is just the Bergman space $A^{\alpha 2}$. Such transforms $W^\alpha_{\psi^{\alpha_0}}$ were also considered by Unterberger in [13], [14] for the quantizations of the Cartan domain $BDI(q=2)$.

Let us now give the expression of the Bergman kernel $K^\alpha_{\psi^\alpha_0}(z,z')$ from (4.5). We have

$$\begin{aligned}
K^\alpha_{\psi^\alpha_0}(z,z') &= C^{-1}_{\psi^\alpha_0} r(y')^{-\frac{\alpha+n+1}{2}} W^\alpha_{\psi^\alpha_0}\left(U_{z'}\psi^\alpha_0\right)(z)\\
&= C^{-1}_{\psi^\alpha_0} r(y')^{-\frac{\alpha+n+1}{2}} \int_C r(y')^{\frac{n+1}{4}} \widehat{\psi^\alpha_0}\left(r(y')^{\frac{1}{2}\,t}\Lambda_{r',s'}\xi\right) e^{-ix'\xi} r(\xi)^{\frac{2\alpha+n+1}{4}} e^{i\xi z}d\xi\\
&= C^{-1}_{\psi^\alpha_0}\int_C r(\xi)^{\frac{2\alpha+n+1}{2}} e^{i\xi(z-\overline{z'})}d\xi.
\end{aligned}$$

Let $A = -i(z-\overline{z'})$; then $\mathrm{Re}A = y+y' \in C$. To compute the integral

$$I(A) = \int_C r(\xi)^{\frac{2\alpha+n+1}{2}} e^{-A\xi}d\xi,$$

we first assume $\mathrm{Im}A = 0$, i.e. $A \in C$; then for general A, $I(A)$ can be gotten from the holomorphic property of $I(A)$. For $A \in C$, there exists $\Lambda \in SO_0(1,n)$ such that $a\Lambda\omega = A$ with $a^2 = r(A)$. Then, by the invariance of the measure $d\xi/r(\xi)^{\frac{n+1}{2}}$ under $\mathbb{R}^*_+ \times SO_0(1,n)$,

$$\begin{aligned}
&\int_C r(\xi)^{\frac{2\alpha+n+1}{2}} e^{-A\xi}d\xi = \int_C r(\xi)^{\frac{2\alpha+n+1}{2}} e^{-[A,\xi]}d\xi\\
&\qquad\text{(with a change of variables } \xi \leftrightarrow J\xi)\\
&= \int_C r(\xi)^{\frac{2\alpha+n+1}{2}} e^{-a[\Lambda\omega,\xi]}d\xi = \int_C r(\xi)^{\frac{2\alpha+n+1}{2}} e^{-a[\omega,\Lambda^{-1}\xi]}d\xi\\
&= \int_C r(\xi)^{\alpha+n+1} e^{-a[\omega,\Lambda^{-1}\xi]}d\xi/r(\xi)^{\frac{n+1}{2}} = \int_C r(\frac{\xi}{a})^{\alpha+n+1} e^{-[\omega,\xi]}d\xi/r(\xi)^{\frac{n+1}{2}}\\
&= a^{-2(\alpha+n+1)}\int_C r(\xi)^{\frac{2\alpha+n+1}{2}} e^{-\xi_0}d\xi = c_n/r(A)^{\alpha+n+1}.
\end{aligned}$$

And therefore

Proposition 4.1. *Let $K(z,z)$ be the Bergman kernel of the tube domain Π. Then*

$$K(z,z') = C_n/r(i(\overline{z'}-z))^{\alpha+n+1}, \tag{4.8}$$

where C_n is a constant.

Π is an (unbounded) realization of $BDI(q=2)$ or the classical bounded domain of type IV (see [15]), and by a transform the kernel given by (4.8) it indeed coincides with the Bergman kernel given by Hua for the case $\alpha = 0$ (see [6, p.88]).

In the rest of this section we want to construct $\psi_{\vec{k}} \in AAW$ such that $A_{\vec{k}} := A^\alpha_{\psi_{\vec{k}}}$ given by (4.4) are orthogonal to each other and their sum is $H^{\alpha 2}(\Pi)$ with $A_{\vec{0}} = A^{\alpha 2}$.

From (4.3), we shall construct $\psi_{\vec{k}} \in AAW$ such that

$$\int_C \widehat{\psi}_{\vec{k}}(y)\overline{\widehat{\psi}_{\vec{k}'}(y)}\frac{dy_0dy_1dy_*}{(y_0+y_1)^{n-1}r(y)} = 0, \quad \text{if } \vec{k} \neq \vec{k}',$$

with $\psi_{\vec{0}} = \psi_0^\alpha$ given by (4.6).

Let H_l be the space of all linear combinations of functions of the form $h(r)Y(x)$, where h ranges over the radial functions and Y ranges over the solid spherical harmonics of degree l. Then $L^2(\mathbb{R}^{n-1})$ can be decomposed as the orthogonal sum (see [10, p.151])

$$L^2(\mathbb{R}^{n-1}) = \bigoplus_{l=0}^{\infty} H_l.$$

Every element h_l in H_l can be written in the form $\sum_{j=1}^{a_l} h_{l,j}(r)Y_j^l(x)$, and

$$\int_{\mathbb{R}^{n-1}} |h_l(x)|^2dx = \sum_{j=1}^{a_l}\int_0^\infty |h_{l,j}(r)|^2r^{n-2}dr,$$

where $a_0 = 1, a_1 = n-1$, $a_l = \binom{n+l-2}{l} - \binom{n+l-4}{l-2}$, $l \geq 2$, and $\{Y_j^l(x)\}_{j=1}^{a_l}$ is an orthogonal basis of the space $\mathcal{H}_l$ of surface spherical harmonics of degree l (see [10, p.140]). It is well known for $n = 3$, $a_l = 2$ and $Y_1^l(x) = \frac{\cos l\theta}{\sqrt{\pi}}, Y_2^l(x) = \frac{\sin l\theta}{\sqrt{\pi}}$ with $x = e^{i\theta}$. For $n > 3$, an orthogonal basis Y_j^l of $\mathcal{H}_l$ can be given by the Gegenbauer polynomials; see [16], pp. 457–468.

If $\psi \in AAW$, then for almost all y_0, y_1, $\widehat{\psi}_{y_0,y_1}(y_*) := \widehat{\psi}(y_0,y_1,y_*)$ is a function of $L^2(\mathbb{R}^{n-1})$, and can be written as

$$\widehat{\psi}(y) = \sum_{l=0}^{\infty}\sum_{j=1}^{a_l} r(y)^{\frac{2\alpha+n+1}{4}} f_{j,l}(y_0,y_1)h_{j,l}\Big(\frac{|y_*|^2}{y_0^2-y_1^2}\Big)Y_j^l(y_*). \tag{4.9}$$

By the orthogonality of Y_j^l, we have

$$\begin{aligned}\int_C |\widehat{\psi}(y)|^2\frac{dy_0dy_1dy_*}{(y_0+y_1)^{n-1}r(y)} &= \sum_{l=0}^{\infty}\sum_{j=1}^{a_l}\int_{y_0>0,y_0^2-y_1^2>r^2}(y_0^2-y_1^2-r^2)^{\alpha+\frac{n-1}{2}} \\ &\quad \cdot |f_{j,l}(y_0,y_1)h_{j,l}\Big(\frac{r^2}{y_0^2-y_1^2}\Big)|^2\frac{r^{n-2}dy_0dy_1dr}{(y_0+y_1)^{n-1}} \\ &= \frac{1}{2}\sum_{l=0}^{\infty}\sum_{j=1}^{a_l}\int_{y_0>|y_1|}|f_{j,l}(y_0,y_1)|^2\frac{(y_0^2-y_1^2)^{\alpha+n-1}}{(y_0+y_1)^{n-1}}dy_0dy_1 \\ &\quad \cdot \int_0^1 |h_{j,l}(t)|^2(1-t)^{\alpha+\frac{n-1}{2}}t^{\frac{n-3}{2}}dt \quad \Big(\text{with } t = \frac{r^2}{y_0^2-y_1^2}\Big)\end{aligned}$$

Let $\vec{k} = (m,\nu,k,l,j)$ and let $\psi_{\vec{k}} \in AAW$ be defined by

$$\widehat{\psi}_{\vec{k}}(y) = r(y)^{\frac{2\alpha+n+1}{4}}e^{-y_0}L_{m,\nu}(y_0,y_1)P_k\Big(\frac{|y_*|^2}{y_0^2-y_1^2}\Big)Y_j^l(y_*),$$

where $L_{m,\nu}, P_k$ are polynomials of degree $m+\nu$ and k respectively. Then from the above calculation and the orthogonality of Y_j^l, we have for $\vec{k}, \vec{k}'$ with $l = l', j = j'$,

(4.10)
$$\begin{aligned}
&\int_C \widehat{\psi}_{\vec{k}}(y)\overline{\widehat{\psi}_{\vec{k}'}(y)} \frac{dy_0 dy_1 dy_*}{(y_0+y_1)^{n-1} r(y)} = C\int_{y_0>|y_1|} L_{m,\nu}(y_0,y_1) L_{m',\nu'}(y_0,y_1) e^{-2y_0} \\
&\quad \cdot \frac{(y_0^2-y_1^2)^{\alpha+n-1}}{(y_0+y_1)^{n-1}} dy_0 dy_1 \int_0^1 P_k(t) P_{k'}(t)(1-t)^{\alpha+\frac{n-1}{2}} t^{\frac{n-3}{2}} dt \\
&= C\int_0^\infty \int_0^\infty L_{m,\nu}(\frac{s+r}{2}, \frac{s-r}{2}) L_{m',\nu'}(\frac{s+r}{2}, \frac{s-r}{2}) e^{-s-r} s^\alpha r^{\alpha+n-1} ds dr \cdot \\
&\quad \cdot \int_0^1 P_k(t) P_{k'}(t)(1-t)^{\alpha+\frac{n-1}{2}} t^{\frac{n-3}{2}} dt \qquad (\text{with } s = \frac{y_0+y_1}{2}, r = \frac{y_0-y_1}{2}).
\end{aligned}$$

Choose

$$\begin{aligned}
L_{m,\nu}(y_0,y_1) &= L_m^{(\alpha)}(y_0+y_1) L_\nu^{(\alpha+n-1)}(y_0-y_1), \\
P_k(t) &= P_k^{(\alpha+\frac{n-1}{2}, \frac{n-3}{2})}(2t-1),
\end{aligned}$$

where $L_k^{(\alpha)}, P_l^{(\alpha,\beta)}$ are the Laguerre polynomials of degree k and the Jacobi polynomials of degree l respectively; then, by the orthogonality of $L_k^{(\alpha)}, P_l^{(\alpha,\beta)}$ (see [11]) and Y_j^l,

$$\begin{aligned}
&\int_C \widehat{\psi}_{\vec{k}}(y)\overline{\widehat{\psi}_{\vec{k}'}(y)} \frac{dy_0 dy_1 dy_*}{(y_0+y_1)^{n-1} r(y)} = C_{k,l,j}\delta_{kk'}\delta_{ll'}\delta_{jj'} \int_0^\infty L_m^{(\alpha)}(s) L_{m'}^{(\alpha)}(s) e^{-s} s^\alpha ds \\
&\quad \cdot \int_0^\infty L_\nu^{(\alpha+n-1)}(r) L_{\nu'}^{(\alpha+n-1)}(r) e^{-r} r^{\alpha+n-1} dr = C_{\vec{k}} \delta_{\vec{k}\vec{k}'}.
\end{aligned}$$

Finally we give a series of orthogonal wavelets

$$\begin{aligned}
\widehat{\psi}_{\vec{k}}(y) =& r(y)^{\frac{2\alpha+n+1}{4}} e^{-y_0} L_m^{(\alpha)}(y_0+y_1) L_\nu^{(\alpha+n-1)}(y_0-y_1) \\
&\cdot P_k^{(\alpha+\frac{n-1}{2}, \frac{n-3}{2})}\left(\frac{2|y_*|^2}{y_0^2-y_1^2} - 1\right) Y_j^l(y_*),
\end{aligned} \tag{4.11}$$

and we have $\psi_{\vec{0}} = \psi_0^\alpha$ given by (4.6). From the completeness properties of the sets $\{L_k^{(\alpha)}(t)e^{-\frac{t}{2}}\}_k$ and $\{P_k^{(\alpha,\beta)}\}_k$ for spaces $L^2(\mathbb{R}_+^*, s^\alpha ds)$, $L^2([-1,1], (1-t)^\alpha(1+t)^\beta dt)$ respectively, we know that $f_{j,l}(y_0,y_1)$, $h_{j,l}\left(\frac{2|y_*|^2}{y_0^2-y_1^2}\right)$ in (4.9) can be written to be orthogonal sums of

$$\{L_m^{(\alpha)}(y_0+y_1)e^{-\frac{y_0+y_1}{2}} \cdot L_\nu^{(\alpha+n-1)}(y_0-y_1)e^{-\frac{y_0-y_1}{2}}\}_{m,\nu}$$

and

$$\{P_k^{(\alpha+\frac{n-1}{2}, \frac{n-3}{2})}\left(\frac{2|y_*|^2}{y_0^2-y_1^2} - 1\right)\}_k$$

respectively. Thus $\{\psi_{\vec{k}}\}_{\vec{k}}$ gives an orthogonal basis of AAW (here and in the following orthogonality of $\psi_{\vec{k}}, \psi_{\vec{k}'}$ means that $\widehat{\psi}_{\vec{k}}, \widehat{\psi}_{\vec{k}'}$ are orthogonal to each other with respect to the measure $dy_0 dy_1 dy_*/(y_0+y_1)^{n-1} r(y)$).

Let $A_{\vec{k}}$ denote the subspaces of $H^{\alpha 2}(\Pi)$ defined by (4.4) with $\psi = \psi_{\vec{k}}$; then the $A_{\vec{k}}$ are orthogonal to each other and in fact they give an orthogonal decomposition of $H^{\alpha 2}(\Pi)$:

Theorem 4.1. *Let $A_{\vec{k}}$ be the subspaces of $H^{\alpha 2}(\Pi)$ defined above. Then*

$$H^{\alpha 2}(\Pi) = \bigoplus_{\vec{k}} A_{\vec{k}},$$

with the first component $A_{\vec{0}} = A^{\alpha 2}$.

The proof of Theorem 4.1 will be given in the Appendix. In [7], we gave two orthogonal bases for AAW in the case $n = 1$, and $\{\psi_{\vec{k}}\}_{\vec{k}}$ given by (4.11) for $n = 1$ is just the suitable basis in [7] for studying the Toeplitz-Hankel type operators between $A_{\vec{k}}$. The orthogonal basis and the decomposition given here will be appropriate for the purpose of studying Toeplitz-Hankel type operators between $A_{\vec{k}}$ for $n \geq 2$.

Appendix

A.1. Admissibility condition in §3. Let us calculate here the admissibility condition for the wavelet transform defined by (3.10). By the Plancherel formula,

$$\begin{aligned}
\int_{G_a} |(\psi, U_g\psi)|^2 d\mu(g) &= \int_{G_a} |\int \hat{\psi}(\xi)\overline{(U_g\psi)\hat{}(\xi)}d\xi|^2 d\mu(g) \\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n+1}} |\int \widehat{\psi}(\xi) a^{\frac{n+1}{2}} \overline{\widehat{\psi}(a^t\Lambda_{r,s}\xi)} e^{ix\xi} d\xi|^2 \frac{dxdsdrda}{a^{n+2}} \\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n+1}} |\widehat{\psi}(\xi)\widehat{\psi}(a^t\Lambda_{r,s}\xi)|^2 \frac{d\xi dsdrda}{a} \\
&= \int_C \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}(a^t\Lambda_{r,s}\xi)|^2 \frac{dsdrda}{a} |\widehat{\psi}(\xi)|^2 d\xi.
\end{aligned}$$

Similarly to the correspondence between AN and H_1 given by (3.3), we also can give a correspondence between ${}^t(AN)$ and H_1. In fact for any $\xi \in C$, there exist unique $a_{r_0} \in A$ and $n_{s_0} \in N$ such that

$${}^t\Lambda_{r_0,s_0}\omega = {}^t n_{s_0} a_{r_0}\omega = \xi / r(\xi)^{\frac{1}{2}},$$

with $e^{r_0} = (\xi_o + \xi_1)/r(\xi)^{\frac{1}{2}}$, $s_0 = {}^t(\xi_2, \cdots, \xi_n)/(\xi_o + \xi_1)$. Since $dsdrda/a$ is invariant under AN, then

(A.0)

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}(a^t\Lambda_{r,s}\xi)|^2 \frac{dsdrda}{a} &= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}(a^t(\Lambda_{r_0,s_0}\Lambda_{r,s})\omega)|^2 \frac{dsdrda}{a} \\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}(a^t\Lambda_{r,s}\omega)|^2 \frac{dsdrda}{a}.
\end{aligned}$$

Thus the admissibility condition is

$$0 < C_\psi = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}(a^t\Lambda_{r,s}\omega)|^2 \frac{dsdrda}{a} < \infty. \tag{A.1}$$

Taking the change of variables

$$\begin{cases} a = r(y)^{\frac{1}{2}}, \\ ae^{-r} = y_0 - y_1, \\ as = y_* = {}^t(y_2, \cdots, y_n), \end{cases} \tag{A.2}$$

with $(a, r, s) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_0, y_1, y_*) \in C$, by a direct calculation

$$dadrds = a^{-n} dy_0 dy_1 dy_* = r(y)^{-\frac{n}{2}} dy_0 dy_1 dy_*, \tag{A.3}$$

and

$$a\Lambda_{r,s}\omega = {}^t\,(a(chr + e^r\rho), a(shr + e^r\rho), as) = {}^t(y_0, y_1, y_*),$$

where $\Lambda_{r,s}$ is given by (3.4) with $\rho = \frac{1}{2}|s|^2$. Thus for a function f on C, we have

$$\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(a\Lambda_{r,s}\omega)\frac{dsdrda}{a} = \int_C f(y)\frac{dy}{r(y)^{\frac{n+1}{2}}}. \tag{A.4}$$

From the definition of $\Lambda_{r,s}$ in (3.4) and the change of variables by (A.2), we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}(a^t\Lambda_{r,s}\omega)|^2\frac{dsdrda}{a}\\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}\,(a(chr + e^r\rho), a(shr - e^r\rho), ae^r s)\,|^2\frac{dsdrda}{a}\\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}\left(a(chr + e^{-r}\rho), a(shr - e^{-r}\rho), as\right)|^2\frac{dsdrda}{e^{r(n-1)}a}\\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{\psi}\,(a(chr + e^r\rho), -a(shr + e^r\rho), as)\,|^2\frac{e^{r(n-1)}dsdrda}{a}\\
&= \int_C |\widehat{\psi}(y_0, -y_1, y_*)|^2\left(\frac{r(y)^{\frac{1}{2}}}{y_0 - y_1}\right)^{n-1}\frac{dy}{r(y)^{\frac{n+1}{2}}} \quad \text{(by (A.4))}\\
&= \int_C |\widehat{\psi}(y_0, y_1, y_*)|^2\frac{dy_0dy_1dy_*}{(y_0 + y_1)^{n-1}r(y)}.
\end{aligned}$$

Thus from (A.1) and above calculation, we finally get the admissibility condition

$$0 < C_\psi = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\widehat{\psi}(y_0, y_1, y_*)|^2\frac{dy_0dy_1dy_*}{(y_0 + y_1)^{n-1}r(y)} < +\infty. \diamond$$

A.2. Proof of Theorem 3.1 and (3.15). The proof of Theorem 3.1 goes like the above calculation of the admissibility condition. In fact, by the Plancherel formula, the change of variables given in (A.2), and formula (A.4), we have

$$\begin{aligned}
&\langle W_\psi f, W_\phi h\rangle = \int_\Pi (W_\psi f)^\wedge(\xi, y)\overline{(W_\phi h)^\wedge(\xi, y)}\frac{d\xi dy}{r(y)^{n+1}}\\
&= \int_C \int_C \widehat{f}(\xi)\overline{\widehat{\psi}(r(y)^{\frac{1}{2}}\,{}^t\Lambda_{r,s}\xi)\widehat{h}(\xi)}\widehat{\phi}(r(y)^{\frac{1}{2}}\,{}^t\Lambda_{r,s}\xi)\frac{d\xi dy}{r(y)^{\frac{n+1}{2}}}\\
&= \int_C \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \overline{\widehat{\psi}(a^t\Lambda_{r,s}\xi)}\widehat{\phi}(a^t\Lambda_{r,s}\xi)\frac{dsdrda}{a}\widehat{f}(\xi)\overline{\widehat{h}(\xi)}d\xi\\
&\qquad \text{(from the change of variables by (A.2))}\\
&= \int_C \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \overline{\widehat{\psi}(a^t\Lambda_{r,s}\omega)}\widehat{\phi}(a^t\Lambda_{r,s}\omega)\frac{dsdrda}{a}\widehat{f}(\xi)\overline{\widehat{h}(\xi)}d\xi\\
&\qquad \text{(similarly to (A.0))}\\
&= \int_C \overline{\widehat{\psi}(y_0, y_1, y_*)}\widehat{\phi}(y_0, y_1, y_*)\frac{dy_0dy_1dy_*}{(y_0 + y_1)^{n-1}r(y)}(f, h)\\
&= (K^{-\frac{1}{2}}\phi, K^{-\frac{1}{2}}\psi)(f, h). \diamond
\end{aligned}$$

Proof of (3.15). Denote $E_{A,B,\delta_1,\delta_2} := \{(x,y) \in \Pi : |x| < A, \delta_1 < y_0 < B, \delta_2 < r(y)/y_0^2\}$. Then we have

$$\begin{aligned}
&\|f - C_\psi^{-1} \int_{E_{A,B,\delta_1,\delta_2}} W_\psi f(z) U_z \psi d\mu(z)\|_2 \\
&\quad = \sup_{\|g\|_2=1, g\in H^2} \left| \left(f - C_\psi^{-1} \int_{E_{A,B,\delta_1,\delta_2}} W_\psi f(z) U_z\psi, \quad g \right) \right| \\
&\quad \le \sup_{\|g\|_2=1, g\in H^2} \left| C_\psi^{-1} \int_{\Pi\setminus E_{A,B,\delta_1,\delta_2}} W_\psi f(z) \overline{W_\psi g(z)} d\mu(z) \right| \\
&\quad \le \sup_{\|g\|_2=1, g\in H^2} \left| C_\psi^{-1} \int_{\Pi\setminus E_{A,B,\delta_1,\delta_2}} |W_\psi f(z)|^2 d\mu(z) \right|^{\frac{1}{2}} \left| C_\psi^{-1} \int_{\Pi} |W_\psi g(z)|^2 d\mu(z) \right|^{\frac{1}{2}} \\
&\quad \le \left| C_\psi^{-1} \int_{\Pi\setminus E_{A,B,\delta_1,\delta_2}} |W_\psi f(z)|^2 d\mu(z) \right|^{\frac{1}{2}} \quad \text{(by Theorem 3.1).}
\end{aligned}$$

Since the infinite integral $\int_\Pi |W_\psi f(z)|^2 d\mu(z)$ converges (by Theorem 3.1 again),

$$\int_{\Pi\setminus E_{A,B,\delta_1,\delta_2}} |W_\psi f(z)|^2 d\mu(z) \to 0, \qquad \text{as } \delta_1, \delta_2 \to 0^+,\ A, B \to +\infty. \ \diamond$$

A.3. Proof of Theorem 4.1. We have shown that $\{C_{\vec{k}}^{-\frac{1}{2}} \psi_{\vec{k}}\}_{\vec{k}}$ is an orthonormal basis of AAW; let $\{\psi_{\vec{k}}\}_{\vec{k}}$ denote this orthonormal basis for simplicity. Let $\{f_\kappa\}_\kappa$ be an orthonormal basis of H^2. In order to prove Theorem 4.1, we need only to show that $\{W^\alpha_{\psi_{\vec{k}}} f_\kappa\}_{\vec{k},\kappa}$ is an orthonormal basis of $H^{2\alpha}(\Pi)$. The orthonormality of these functions is obvious (from (4.3)) and we need to show their completeness (closure) in $H^{2\alpha}(\Pi)$, for which it is enough to show that

$$\|F\|^2_{L^{2\alpha}(\Pi)} = \sum_{\vec{k},\kappa} |(F, W^\alpha_{\psi_{\vec{k}}} f_\kappa)_{L^{2\alpha}(\Pi)}|^2 \tag{A.5}$$

for any $F \in H^{2\alpha}(\Pi)$. We have for $F \in H^{2\alpha}(\Pi)$

$$\begin{aligned}
&\int_\Pi F(x,y) \overline{W^\alpha_{\psi_{\vec{k}}} f_\kappa(x,y)} r(y)^\alpha dxdy \\
&= \int_C \int_C \widehat{F}(\xi, y) \overline{\widehat{f_\kappa}(\xi)} r(y)^{-\frac{2\alpha+n+1}{4}} \widehat{\psi}_{\vec{k}}(r(y)^{\frac{1}{2}t} \Lambda_{r,s}\xi) r(y)^\alpha d\xi dy \\
&\qquad (\text{with } \Lambda_{r,s}\omega = y/r(y)^{\frac{1}{2}}) \\
&= \int_C \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \overline{\widehat{f_\kappa}(\xi)} \widehat{F}(\xi, a\Lambda_{r,s}\omega) a^{\frac{2\alpha+n+1}{2}} \widehat{\psi}_{\vec{k}}(a^t \Lambda_{r,s}\xi) \frac{dadrds}{a} d\xi \\
&\qquad (\text{from the change of variables by (A.2)}) \\
&= \int_C \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \overline{\widehat{f_\kappa}(\xi)} \widehat{F}(\xi, a\Lambda_{r,s}\omega) \widehat{\psi}_{\vec{k}} \left(ar(\xi)^{\frac{1}{2}t} (\Lambda_{r_0,s_0}\Lambda_{r,s})\omega \right) \\
&\qquad \cdot a^{\frac{2\alpha+n+1}{2}} \frac{dadrds}{a} d\xi \quad (\text{with } \xi = r(\xi)^{\frac{1}{2}t} \Lambda_{r_0,s_0}\omega)
\end{aligned}$$

$$= \int_C \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \overline{\widehat{f_\kappa}(\xi)} \widehat{F}\left(\xi, \frac{a}{r(\xi)^{\frac{1}{2}}} \Lambda_{r_0,s_0}^{-1} \Lambda_{r,s}\omega\right) \left(\frac{a}{r(\xi)^{\frac{1}{2}}}\right)^{\frac{2\alpha+n+1}{2}}$$
$$\cdot \widehat{\psi}_{\vec{k}}(a^t \Lambda_{r,s}\omega) \frac{dadrds}{a} d\xi$$
$$= \left(M_{\vec{k}}, f_\kappa\right)_{H^2},$$

where the $M_{\vec{k}}$ are given by (for $\xi \in C$)

(A.6)
$$\widehat{M}_{\vec{k}}(\xi) = \frac{1}{r(\xi)^{\frac{2\alpha+n+1}{4}}} \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \widehat{F}\left(\xi, \frac{a}{r(\xi)^{\frac{1}{2}}} \Lambda_{r_0,s_0}^{-1} \Lambda_{r,s}\omega\right) a^{\frac{2\alpha+n+1}{2}} \widehat{\psi}_{\vec{k}}(a^t \Lambda_{r,s}\omega) \frac{dadrds}{a}.$$

Since $\{f_\kappa\}_\kappa$ is an orthonormal basis of H^2, we have

$$\sum_{\vec{k},\kappa} |(F, W^\alpha_{\psi_{\vec{k}}} f_\kappa)_{L^{2\alpha}(\Pi)}|^2 = \sum_{\vec{k},\kappa} |\left(M_{\vec{k}}, f_\kappa\right)|^2 = \sum_{\vec{k}} \|M_{\vec{k}}\|^2_{H^2}. \tag{A.7}$$

By (A.6) and the fact that $\{\widehat{\psi}_{\vec{k}}\}_{\vec{k}}$ is an orthonormal basis of $L^2(C, \frac{d\xi}{(\xi_0+\xi_1)^{n-1} r(\xi)})$ (or $\{\widehat{\psi}_{\vec{k}}(a^t\Lambda_{r,s}\omega)\}_{\vec{k}}$ is an orthonormal basis of $L^2(\mathbb{R}^*_+ \times \mathbb{R}^n, dadrds/a)$) , we have

$$\sum_{\vec{k}} |\widehat{M}_{\vec{k}}(\xi)|^2$$
$$= r(\xi)^{-\frac{2\alpha+n+1}{2}} \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{F}(\xi, \frac{a}{r(\xi)^{\frac{1}{2}}} \Lambda_{r_0,s_0}^{-1} \Lambda_{r,s}\omega)|^2 a^{2\alpha+n+1} \frac{dadrds}{a}$$
$$= r(\xi)^{-\frac{2\alpha+n+1}{2}} \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\widehat{F}(\xi, a\Lambda_{r,s}\omega)|^2 r(\xi)^{\frac{2\alpha+n+1}{2}} a^{2\alpha+n+1} \frac{dadrds}{a}$$
$$= \int_C |\widehat{F}(\xi, y)|^2 r(y)^{\frac{2\alpha+n+1}{2}} \frac{dy}{r(y)^{\frac{n+1}{2}}} \quad \text{(by (A.4) again)}$$
$$= \int_C |\widehat{F}(\xi, y)|^2 r(y)^\alpha dy,$$

and thus

$$\sum_{\vec{k}} \|M_{\vec{k}}\|^2_{H^2} = \sum_{\vec{k}} \int_C |\widehat{M}_{\vec{k}}(\xi)|^2 d\xi \tag{A.8}$$
$$= \int_C \int_C |\widehat{F}(\xi, y)|^2 r(y)^\alpha dy d\xi = \int_\Pi |F(x, y)|^2 r(y)^\alpha dx dy.$$

From (A.7) and (A.8), we get (A.5), and the proof of Theorem 4.2 is completed.$\diamond$

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